Recursive boson system in the Cuntz algebra $\mathcal{O}_\infty$

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Abstract

Bosons and fermions are often written by elements of other algebras. M. Abe gave a recursive realization of the boson by formal infinite sums of the canonical generators of the Cuntz algebra $\mathcal{O}_\infty$. We show that such formal infinite sum always makes sense on a certain dense subspace of any permutative representation of $\mathcal{O}_\infty$. In this meaning, we can regard as if the algebra $\mathcal{B}$ of bosons was a unital $*$-subalgebra of $\mathcal{O}_\infty$ on a given permutative representation by keeping their unboundedness. By this relation, we compute branching laws arising from restrictions of representations of $\mathcal{O}_\infty$ on $\mathcal{B}$. For example, it is shown that the Fock representation of $\mathcal{B}$ is given as the restriction of the standard representation of $\mathcal{O}_\infty$ on $\mathcal{B}$.

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1 Introduction

Bosons and fermions are not only important in physics but also interesting in mathematics. Studies of their algebras spurred the development of the theory of operator algebras [8]. Representations of bosons are used to describe representations of several algebras [6, 12, 15]. Bosons and fermions are often written by elements of other algebras and such descriptions are useful for several computations. For example, the boson-fermion correspondence [16, 17] is well-known. It is shown that bosons and fermions are corresponded as operators on the infinite wedge representation of fermions.

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1.1 Motivation

In our previous paper [1], we have presented a recursive construction of the CAR (=canonical anticommutation relation) algebra for fermions in terms of the Cuntz algebra $\mathcal{O}_2$ and shown that it may provide us a useful tool to study properties of fermion systems by using explicit expressions in terms of generators of the algebra. Let s_1, s_2 be the canonical generators of $\mathcal{O}_2$, that is, they satisfy that

$$s_i^* s_j = \delta_{ij} I \quad (i, j = 1, 2), \quad s_1 s_1^* + s_2 s_2^* = I.$$

Let ζ be the linear map on $\mathcal{O}_2$ defined by $\zeta(x) \equiv s_1 x s_1^* - s_2 x s_2^*$ for $x \in \mathcal{O}_2$. We recursively define the family $\{a_1, a_2, a_3, \ldots\}$ by

$$a_1 \equiv s_1 s_2^*, \quad a_n \equiv \zeta(a_{n-1}) \quad (n \geq 2).$$

Then $\{a_n : n \in \mathbf{N}\}$ satisfies that

$$a_n a_m^* + a_m^* a_n = \delta_{nm} I, \quad a_n a_m + a_m a_n = a_n^* a_m^* + a_m^* a_n^* = 0 \quad (n, m \in \mathbf{N})$$

where $\mathbf{N} = \{1, 2, 3, \ldots\}$. We call such $\{a_n : n \in \mathbf{N}\}$ by a *recursive fermion system (=RFS)* in $\mathcal{O}_2$. From this description, the C*-algebra $\mathcal{A}$ generated by fermions is embedded into $\mathcal{O}_2$ as a C*-subalgebra with common unit:

$$\mathcal{A} \equiv C^* \langle \{a_n : n \in \mathbf{N}\} \rangle \hookrightarrow \mathcal{O}_2$$

Furthermore $\mathcal{A}$ coincides with the fixed-point subalgebra of $\mathcal{O}_2$ with respect to the $U(1)$-gauge action. Because every a_n is written as a polynomial in the canonical generators of $\mathcal{O}_2$ and their $*$-conjugates, their description is very simple and it is easy to compute the restriction $\pi|_{\mathcal{A}}$ of a representation π of $\mathcal{O}_2$ on $\mathcal{A}$. By using the RFS, we obtain several new results about fermions [2, 3, 4, 5]. For example, assume that $(\mathcal{H}, \pi)$ is a $*$-representation of $\mathcal{O}_2$ with a cyclic vector Ω. If Ω satisfies $\pi(s_1)\Omega = \Omega$, then $\pi|_{\mathcal{A}}$ is equivalent to the Fock representation of $\mathcal{A}$ with the vacuum Ω. If Ω satisfies $\pi(s_1 s_2)\Omega = \Omega$, then $\pi|_{\mathcal{A}}$ is equivalent to the direct sum of the infinite wedge representation and the dual infinite wedge representation of $\mathcal{A}$ [14]. In this way, well-known results of fermions are explicitly reformulated by the representation theory of $\mathcal{O}_2$.

From this, we speculate that the boson can be also simply written by the generators of a certain Cuntz algebra like the RFS, where the boson means a family $\{a_n : n \in \mathbf{N}\}$ satisfying that

$$a_n a_m^* - a_m^* a_n = \delta_{nm} I, \quad a_n a_m - a_m a_n = a_n^* a_m^* - a_m^* a_n^* = 0 \tag{1.1}$$

for each $n, m \in \mathbf{N}$. However, the boson is always represented as a family of unbounded operators on a Hilbert space. Hence the $*$-algebra generated by $\{a_n : n \in \mathbf{N}\}$ never be a $*$-subalgebra of any C*-algebra. On the other hand, the C*-algebra approach of boson is well-known as the CCR algebra (CCR = canonical commutation relations, see § 5.2 in [8]). Because the CCR algebra is not a separable C*-algebra, it is impossible to embed it into any Cuntz algebra as a C*-subalgebra. From these problems, it seems that a RFS-like description of bosons by any Cuntz algebra is impossible.

1.2 Recursive boson system

In spite of such problems, Mitsuo Abe gave a "formal" realization of the boson by the canonical generators of the Cuntz algebra $\mathcal{O}_\infty$ in 2006 as follows. Let $\{s_n : n \in \mathbf{N}\}$ be the canonical generators of $\mathcal{O}_\infty$, that is,

$$s_i^* s_j = \delta_{ij} I \quad (i, j \in \mathbf{N}), \quad \sum_{i=1}^{k} s_i s_i^* \leq I \quad (\text{for any } k \in \mathbf{N}).$$

Define the family $\{a_n : n \in \mathbf{N}\}$ of formal sums by

$$a_1 \equiv \sum_{m=1}^{\infty} \sqrt{m}\, s_m s_{m+1}^*, \quad a_n \equiv \rho(a_{n-1}) \quad (n \geq 2) \tag{1.2}$$

where ρ is the formal canonical endomorphism of $\mathcal{O}_\infty$ defined by

$$\rho(x) \equiv \sum_{n=1}^{\infty} s_n x s_n^* \quad (x \in \mathcal{O}_\infty).$$

By formal computation, we can verify that a_n's satisfy (1.1) where we assume that infinite sums can be freely exchanged. However infinite sums in these equations *do not* converge in $\mathcal{O}_\infty$ in general. Hence (1.2) does not make sense as elements of $\mathcal{O}_\infty$.

We show that the Abe's formal description (1.2) can be justified as unbounded operators defined on a certain dense subspace of any permutative representation of $\mathcal{O}_\infty$. Define $X_N \equiv \{1, \ldots, N\}$ for $2 \leq N < \infty$ and $X_\infty \equiv \mathbf{N}$. Let $\{s_n : n \in X_N\}$ be the set of canonical generators of $\mathcal{O}_N$ for $2 \leq N \leq \infty$.

Definition 1.1 *[7, 10, 11] A representation $(\mathcal{H}, \pi)$ of $\mathcal{O}_N$ is permutative if there exists a complete orthonormal basis $\{e_n\}_{n\in\Lambda}$ of $\mathcal{H}$ and a family $f = \{f_i\}_{i=1}^N$ of maps on Λ such that $\pi(s_i)e_n = e_{f_i(n)}$ for each $n \in \Lambda$ and $i = 1, \ldots, N$. We call $\{e_n\}_{n\in\Lambda}$ and the linear hull $\mathcal{D}$ of $\{e_n\}_{n\in\Lambda}$ by the reference basis and the reference subspace of $(\mathcal{H}, \pi)$, respectively.*

Remark that for any permutation representation $(\mathcal{H},\pi)$ of $\mathcal{O}_N$ with the reference subspace $\mathcal{D}$, $\pi(s_n)\mathcal{D}\subset\mathcal{D}$ and $\pi(s_n^*)\mathcal{D}\subset\mathcal{D}$ for each n, but $\pi(x)\mathcal{D}\not\subset\mathcal{D}$ for $x\in\mathcal{O}_\infty$ in general.

From Definition 1.1 and (1.2), we immediately obtain the following fact.

Fact 1.2 *For any permutative representation $(\mathcal{H},\pi)$ of $\mathcal{O}_\infty$, define the family $\{A_n : n\in\mathbf{N}\}$ of operators on the reference subspace $\mathcal{D}$ of $(\mathcal{H},\pi)$ by*

$$A_1 v \equiv \sum_{m=1}^{\infty}\sqrt{m}\,\pi(s_m s_{m+1}^*)v,$$

$$A_n v \equiv \sum_{m_{n-1}=1}^{\infty}\cdots\sum_{m_1=1}^{\infty}\sum_{m=1}^{\infty}\sqrt{m}\,\pi(s_{m_{n-1}}\cdots s_{m_1}s_m s_{m+1}^* s_{m_1}^*\cdots s_{m_{n-1}}^*)v$$

for $v\in\mathcal{D}$ and $n\geq 2$. Then the family $\{A_n : n\in\mathbf{N}\}$ satisfies (1.1) on $\mathcal{D}$.

Infinite sums in Fact 1.2 are actually finite for each $v\in\mathcal{D}$. By comparing Fact 1.2 and (1.2), we see that (1.2) is well-defined on the reference subspace of any permutative representation of $\mathcal{O}_\infty$. Furthermore, the mapping

$$a_n\mapsto A_n\quad(n\in\mathbf{N})\tag{1.3}$$

defines a unital $*$-representation $\pi_{\mathcal{B}}$ of the algebra $\mathcal{B}$ of bosons on $\mathcal{D}$, that is, $\pi_{\mathcal{B}}(a_n)\equiv A_n$ for each n. In consequence, we obtain the operation

$$(\mathcal{H},\pi)\mapsto(\mathcal{D},\pi_{\mathcal{B}})$$

for any permutative representation $(\mathcal{H},\pi)$ of $\mathcal{O}_\infty$ to the representation $(\mathcal{D},\pi_{\mathcal{B}})$ of $\mathcal{B}$. We call $(\mathcal{D},\pi_B)$ the *restriction of* $(\mathcal{H},\pi)$ on $\mathcal{B}$ and often write it by $(\mathcal{H},\pi|_{\mathcal{B}})$ for convenience in this paper. Strictly speaking, this is not a restriction because $\mathcal{B}$ is neither a subalgebra of $\mathcal{O}_\infty$ nor $\pi(\mathcal{O}_\infty)\mathcal{D}\subset\mathcal{D}$.

Remark 1.3 If a C*-algebra $\mathcal{A}$ irreducibly acts on a Hilbert space $\mathcal{H}$, then any (unbounded) operator on $\mathcal{H}$ can be written by the strong operator limit of elements of $\mathcal{A}$ on $\mathcal{H}$. However such description always depends on the choice of representation. Fact 1.2 claims that the description (1.2) *always* hold on *any* permutative representations of $\mathcal{O}_\infty$ nevertheless there exist infinitely many inequivalent permutative representations of $\mathcal{O}_\infty$ and they are not always irreducible.

Definition 1.4 *The family $\{A_n : n\in\mathbf{N}\}$ in Fact 1.2 is called the recursive boson system (=RBS) in $\mathcal{O}_\infty$ with respect to a permutative representation $(\mathcal{H},\pi)$ of $\mathcal{O}_\infty$.*

We identify A_n in Fact 1.2 with a_n.

Remark that $\mathcal{B}$ is neither a subalgebra of $\mathcal{O}_\infty$ nor that of the double commutations $\pi(\mathcal{O}_\infty)''$ of $\pi(\mathcal{O}_\infty)$. However for any permutative representation $(\mathcal{H}, \pi)$ of $\mathcal{O}_\infty$, we obtain a representation of the boson as $\mathcal{B}$ by the RBS. In this sense, it seems that $\mathcal{B}$ is a subalgebra of $\mathcal{O}_\infty$ in special situation:

$$\mathcal{B} = \mathrm{Alg}\langle\{a_n, a_n^* : n \in \mathbf{N}\}\rangle \quad \risingdotseq \quad \text{subalgebra of } \mathcal{O}_\infty.$$

1.3 Representations of bosons arising from permutative representations of $\mathcal{O}_\infty$

We show the significance of the RBS in the representation theory of operator algebras. The algebra $\mathcal{B}$ of bosons always appears with a representation in theoretical physics. Especially, the Fock representation plays the most important role among representations of $\mathcal{B}$. It has both the mathematical simple structure and the physical meaning. By the RBS, we can understand the Fock representation from a viewpoint of the representation theory of $\mathcal{O}_\infty$.

First, we explain the notion of branching law. For a group G, if there exists an embedding of G into some other group G', then any representation π of G' induces the restriction $\pi|_G$ of π on G. The representation $\pi|_G$ is not irreducible in general even if π is irreducible. If $\pi|_G$ is decomposed into the direct sum of a family $\{\pi_\lambda : \lambda \in \Lambda\}$ of irreducible representations of G, then the equation

$$\pi|_G = \bigoplus_{\lambda\in\Lambda} \pi_\lambda$$

is called the *branching law* of π. The branching law can be also considered for a pair of subalgebra and algebra. Thanks to the RBS, we can consider (an analogy of) branching laws of permutative representations of $\mathcal{O}_\infty$ which are restricted on $\mathcal{B}$.

Theorem 1.5 (i) *For $j \geq 1$, let $(\mathcal{H}, \pi_j)$ be a representation of $\mathcal{O}_\infty$ with a cyclic vector Ω satisfying*

$$\pi_j(s_j)\Omega = \Omega.$$

Then there exists a dense subspace $\mathcal{D}_j$ of $\mathcal{H}$ and an action η_i of $\mathcal{B}$ on $\mathcal{D}_j$ such that $\eta_j(\mathcal{B})\Omega = \mathcal{D}_j$ and

$$\eta_j(a_n a_n^*)\Omega = j\Omega \quad (n \geq 1). \tag{1.4}$$

In particular, η_1 is the Fock representation of $\mathcal{B}$ with the vacuum Ω.

(ii) *Let* $(\mathcal{H}, \pi_{12})$ *be a representation of* $\mathcal{O}_\infty$ *with a cyclic vector* Ω *satisfying*

$$\pi_{12}(s_1 s_2)\Omega = \Omega.$$

Then there exist two subspaces V_1 *and* V_2 *of* $\mathcal{H}$ *and two actions* η_{12} *and* η_{21} *of* $\mathcal{B}$ *on* V_1 *and* V_2*, respectively such that* $V_1 \oplus V_2$ *is dense in* $\mathcal{H}$*,* $V_1 = \eta_{12}(\mathcal{B})\Omega$*,* $V_2 = \eta_{21}(\mathcal{B})\Omega'$ *for* $\Omega' \equiv \pi(s_2)\Omega$ *and the following holds:*

$$\begin{cases} \eta_{12}(a_{2n-1})\Omega = \eta_{21}(a_{2n})\Omega' = 0, & \\ \eta_{12}(a_{2n}^* a_{2n})\Omega = \Omega, & (n \geq 1). \\ \eta_{21}(a_{2n-1}^* a_{2n-1})\Omega' = \Omega' & \end{cases}$$

(iii) *Any two of representations in* $\{\eta_j, \eta_{12}, \eta_{21} : j \geq 1\}$ *of* $\mathcal{B}$ *are not unitarily equivalent.*

(iv) *All of representations* $\{\eta_j, \eta_{12}, \eta_{21} : j \geq 1\}$ *of* $\mathcal{B}$ *are irreducible.*

Every representations of $\mathcal{O}_\infty$ in Theorem 1.5 (i) and (ii) are irreducible permutative representations. Hence Theorem 1.5 shows branching laws of representations of $\mathcal{O}_\infty$ restricted on $\mathcal{B}$:

$$\pi_j|_{\mathcal{B}} = \eta_j \quad (j \geq 1), \quad \pi_{12}|_{\mathcal{B}} = \eta_{12} \oplus \eta_{21}.$$

By comparison to the fermion case in § 1.1, this result shows that the RBS is very similar to the RFS in a sense of the representation theory of operator algebras. This result shows the naturality of the description in (1.2).

In § 2, we show permutative representations of $\mathcal{O}_\infty$ and several representations of $\mathcal{B}$. In § 2.3, we prove Theorem 1.5. In § 3, we show examples. In § 3.2, we give an interpretation of representations of bosons in Theorem 1.5 by formal infinite product of operators.

2 Representations and their relations

In order to show Theorem 1.5, we introduce several representations of $\mathcal{O}_\infty$ and $\mathcal{B}$. After this preparation, we show their relations as the proof of Theorem 1.5.

2.1 Permutative representation of Cuntz algebras

For $N = 2, 3, \ldots, +\infty$, let $\mathcal{O}_N$ be the *Cuntz algebra* [9], that is, a C*-algebra which is universally generated by $s_1, \ldots, s_N$ satisfying $s_i^* s_j = \delta_{ij} I$ for $i, j = 1, \ldots, N$ and

$$\sum_{i=1}^{N} s_i s_i^* = I \quad (\text{if } N < +\infty), \quad \sum_{i=1}^{k} s_i s_i^* \leq I, \quad k = 1, 2, \ldots, \quad (\text{if } N = +\infty)$$

where I is the unit of $\mathcal{O}_N$. Because $\mathcal{O}_N$ is simple, that is, there is no non-trivial closed two-sided ideal, any homomorphism from $\mathcal{O}_N$ to a C*-algebra is injective. If $t_1, \ldots, t_n$ are elements of a unital C*-algebra $\mathcal{A}$ such that $t_1, \ldots, t_n$ satisfy the relations of canonical generators of $\mathcal{O}_N$, then the correspondence $s_i \mapsto t_i$ for $i = 1, \ldots, N$ is uniquely extended to a $*$-embedding of $\mathcal{O}_N$ into $\mathcal{A}$ from the uniqueness of $\mathcal{O}_N$. Therefore we call such a correspondence among generators by an embedding of $\mathcal{O}_N$ into $\mathcal{A}$.

Define $X_N \equiv \{1, \ldots, N\}$ for $2 \leq N < \infty$ and $X_\infty \equiv \mathbf{N}$. For $N = 2, \ldots, \infty$ and $k = 1, \ldots, \infty$, define the product set $X_N^k \equiv (X_N)^k$ of X_N. Let $\{s_n : n \in X_N\}$ be the set of canonical generators of $\mathcal{O}_N$ for $2 \leq N \leq \infty$.

Definition 2.1 *For $J = (j_l)_{l=1}^k \in X_N^k$ with $1 \leq k < \infty$, we write $P_N(J)$ the class of representations $(\mathcal{H}, \pi)$ of $\mathcal{O}_N$ with a cyclic unit vector $\Omega \in \mathcal{H}$ such that $\pi(s_J)\Omega = \Omega$ and $\{\pi(s_{j_l} \cdots s_{j_k})\Omega\}_{l=1}^k$ is an orthonormal family in $\mathcal{H}$ where $s_J \equiv s_{j_1} \cdots s_{j_k}$. Here, $\{\pi(s_{j_l} \cdots s_{j_k})\Omega\}_{l=1}^k$ is called the cycle of $P_N(J)$.*

We call the vector Ω in Definition 2.1 by the *GP vector* of $(\mathcal{H}, \pi)$. A representation $(\mathcal{H}, \pi)$ of $\mathcal{O}_N$ is called a *cycle* if there exists $J \in X_N^k$ for $1 \leq k < \infty$ such that $(\mathcal{H}, \pi)$ belongs to $P_N(J)$. Any permutative representation is uniquely decomposed into cyclic permutative representations up to unitary equivalence. For any J, $P_N(J)$ contains only one unitary equivalence class [7, 10, 11, 13]. We show properties of $P_\infty(j)$ $(j \geq 1)$ and $P_\infty(12)$ more closely as follows.

Lemma 2.2 *Let $\mathcal{T} \equiv \{P_\infty(j), P_\infty(12) : j \geq 1\}$.*

(i) *For each $X \in \mathcal{T}$, any two representations belonging to X are unitarily equivalent.*

(ii) *Any two of representations in $\mathcal{T}$ are not unitarily equivalent.*

(iii) *All of representations in $\mathcal{T}$ are irreducible.*

The proof of Lemma 2.2 are given in Appendix A. From Lemma 2.2 (i), we use symbols $P_\infty(j), P_\infty(12)$ as their representatives.

For $2 \leq N < \infty$, let $t_1, \ldots, t_N$ be the canonical generators of $\mathcal{O}_N$. Define the representation $(l_2(\mathbf{N}), \pi)$ of $\mathcal{O}_N$ by

$$\pi(t_i)e_n \equiv e_{N(n-1)+i} \quad (i = 1, \ldots, N,\, n \in \mathbf{N}).$$

Then $(l_2(\mathbf{N}), \pi)$ is $P_N(1)$ of $\mathcal{O}_N$. If we identify $\mathcal{O}_\infty$ with a C*-subalgebra of $\mathcal{O}_N$ by the embedding of $\mathcal{O}_\infty$ into $\mathcal{O}_N$ defined by

$$s_{(N-1)(k-1)+i} \equiv t_N^{k-1} t_i \quad (k \geq 1,\, i = 1, \ldots, N-1), \tag{2.1}$$

then $(l_2(\mathbf{N}), \pi|_{\mathcal{O}_\infty})$ is $P_\infty(1)$ of $\mathcal{O}_\infty$.

2.2 Representations of bosons

We summarize several representations of bosons and their properties. We write $\mathcal{B}$ the $*$-algebra generated by $\{a_n : n \in \mathbf{N}\}$ which satisfies (1.1). A *representation* of $\mathcal{B}$ is a pair $(\mathcal{H}, \pi)$ such that $\mathcal{H}$ is a complex Hilbert space with a dense subspace $\mathcal{D}$ and π is a $*$-homomorphism from $\mathcal{B}$ to the $*$-algebra $\{x \in \mathrm{End}_{\mathbf{C}}(\mathcal{D}) : x^*\mathcal{D} \subset \mathcal{D}\}$. A *cyclic vector* of $(\mathcal{H}, \pi)$ is a vector $\Omega \in \mathcal{D}$ such that $\pi(\mathcal{B})\Omega = \mathcal{D}$.

Definition 2.3 (i) *For $j \geq 1$, we write F_j the class of representations $(\mathcal{H}, \pi)$ of $\mathcal{B}$ with a cyclic vector Ω satisfying $\pi(a_n a_n^*)\Omega = j\Omega$ for each $n \in \mathbf{N}$.*

(ii) *We write F_{12} the class of representations $(\mathcal{H}, \pi)$ of $\mathcal{B}$ with a cyclic vector Ω satisfying*

$$\pi(a_{2n-1})\Omega = 0, \quad \pi(a_{2n}^* a_{2n})\Omega = \Omega$$

for each $n \in \mathbf{N}$.

(iii) *We write F_{21} the class of representations $(\mathcal{H}, \pi)$ of $\mathcal{B}$ with a cyclic vector Ω satisfying*

$$\pi(a_{2n})\Omega = 0, \quad \pi(a_{2n-1}^* a_{2n-1})\Omega = \Omega$$

for each $n \in \mathbf{N}$.

A representation $(\mathcal{H}, \pi)$ of $\mathcal{B}$ is called *irreducible* if the commutant of $\pi(\mathcal{B})$ in $\mathcal{B}(\mathcal{H})$ is the scalar multiples of I.

Lemma 2.4 *Let* $\mathcal{S} \equiv \{F_j, F_{12}, F_{21} : j \geq 1\}$.

(i) *For each* $X \in \mathcal{S}$, *any two representations belonging to* X *are unitary equivalent. From this, we can identify a representation belonging to* $X \in \mathcal{S}$ *with* X.

(ii) *Any two of representations in* $\mathcal{S}$ *are not unitarily equivalent.*

(iii) *All of representations in* $\mathcal{S}$ *are irreducible.*

Lemma 2.4 is proved in Appendix B. We consider the case $j = 1$ in Definition 2.3 (i). Then $\pi(a_n a_n^*)\Omega = \Omega$ for each n. From this, $\pi(a_n^* a_n)\Omega = 0$. This implies that $\pi(a_n)\Omega = 0$ for each n. Because Ω is a cyclic vector, F_1 is the Fock representation of $\mathcal{B}$ with the vacuum Ω.

In this study, we became the first to find F_j, F_{12}, F_{21} from the computation of branching laws of permutative representations of $\mathcal{O}_\infty$. After finding the equations of bosons and the vector Ω, we found the conditions of F_j, F_{12}, F_{21} without using permutative representations of $\mathcal{O}_\infty$.

2.3 Proof of Theorem 1.5

Before the proof, we summarize basic relations of the RBS $\{a_n : n \in \mathbf{N}\}$ and the canonical generators $\{s_n : n \in \mathbf{N}\}$ of $\mathcal{O}_\infty$. From (1.2), the following holds on the reference subspace of any permutative representation of $\mathcal{O}_\infty$:

$$s_m a_n = a_{n+1} s_m, \quad s_m a_n^* = a_{n+1}^* s_m \quad (n, m \in \mathbf{N}),$$

$$\rho(x) s_i = s_i x \quad (x \in \mathcal{O}_\infty,\ i \in \mathbf{N}).$$

(i) Fix $j \geq 1$. First, we see that $(\mathcal{H}, \pi_j)$ is $P_\infty(j)$ with the GP vector Ω. We simply write $\pi_j(s_n)$ by s_n for each n. Define

$$\mathcal{D}_j \equiv \mathrm{Lin}\langle \{s_J \Omega : J \in \mathbf{N}^*\} \rangle$$

where $\mathbf{N}^* \equiv \coprod_{l \geq 1} \mathbf{N}^l$. Then $\mathcal{D}_j$ is the reference subspace. We simply write $\{a_n : n \in \mathbf{N}\}$ the RBS on $P_\infty(j)$ and $\mathcal{B}$ the algebra generated by them. From (1.2),

$$a_n a_n^* = \sum_{K \in \mathbf{N}^{n-1}} \sum_{m=1}^{\infty} m s_K s_m s_m^* s_K^*.$$

By definition, $s_j^m \Omega = (s_j^*)^m \Omega = \Omega$ for any $m \geq 1$. From these, we obtain that $a_n a_n^* \Omega = j\Omega$ for any $n \in \mathbf{N}$.

It is sufficient to show $\mathcal{B}\Omega = \mathcal{D}_j$. By definition of the RBS, $\mathcal{B}\Omega \subset \mathcal{D}_j$. We write $(a_n^*)^{-1} \equiv a_n$ and $a_n^0 = (a_n^*)^0 = I$ for convenience. Then for any $n \in \mathbf{N}$, there exists $M \in \mathbf{R}$ such that $s_n\Omega = M(a_1^*)^{n-j}\Omega$. From this, we can derive that

$$s_K\Omega \in \mathcal{B}\Omega \quad (K \in \mathbf{N}^*).$$

Hence $\mathcal{D}_j \subset \mathcal{B}\Omega$. Therefore the statement holds.
(ii) We see that $(\mathcal{H}, \pi_{12})$ is $P_\infty(12)$ with the GP vector Ω. The relations of a_n's and Ω, Ω' are shown by assumption. Let $V_1 \equiv \mathcal{B}\Omega$ and $V_2 \equiv \mathcal{B}\Omega'$. Then we see that V_1 and V_2 are F_{12} and F_{21}, respectively. By Lemma 2.4 (ii), V_1 and V_2 are orthogonal in $\mathcal{H}$.

For $m \geq 1$ and $J = (j_1, \ldots, j_n) \in \mathbf{N}^n$,

$$s_J\Omega = \begin{cases} C_n a^{*(J-1)} a_2 a_4 \cdots a_{2m}\Omega & (n = 2m), \\ \\ C_n a^{*(J-1)} a_1 a_3 \cdots a_{2m-1}\Omega' & (n = 2m-1) \end{cases}$$

where $a^{*(J-1)} \equiv (a_1^*)^{j_1-1} \cdots (a_n^*)^{j_n-1}$ and $C_n \equiv \{(j_1-1)! \cdots (j_n-1)!\}^{-1/2}$. From this, $s_J\Omega \in V_1 \oplus V_2$ for any $J \in \mathbf{N}^*$. This implies that the reference subspace of $\mathcal{H}$ is a subspace of $V_1 \oplus V_2$. Hence $V_1 \oplus V_2$ is dense in $\mathcal{H}$.
(iii) From (i), (ii) and Lemma 2.4 (i), we see that η_j is F_j $(j \geq 1)$, η_{12} is F_{12} and η_{21} is F_{21}. From these and Lemma 2.4 (ii), the statement holds.
(iv) From Lemma 2.4 (iii), the statement holds. ∎

3 Example

3.1 Fock representation of RBS

From Theorem 1.5 (i), we obtain a correspondence between state vectors in the Bose-Fock space and vectors in the permutative representation $P_\infty(1)$ as follows:

$$(a_1^*)^{j_1-1} \cdots (a_k^*)^{j_k-1}\Omega = \{(j_1-1)! \cdots (j_k-1)!\}^{1/2} s_J\Omega \tag{3.1}$$

for $J = (j_1, \ldots, j_k) \in \mathbf{N}^k$. This shows that any physical theory with the Bose-Fock space is rewritten by $\mathcal{O}_\infty$. Furthermore the Fock vacuum is interpreted as the eigenvector of the generator s_1 of $\mathcal{O}_\infty$. For example, the one-particle state is given as follows:

$$a_n^*\Omega = s_1^{n-1}s_2\Omega \quad (n \geq 1).$$

On the other hand, if the Fock representation of $\mathcal{B}$ is given, then it is always extended to the action of $\mathcal{O}_\infty$ as follows:

$$\begin{aligned}
s_m\Omega =& \ \ \{(m-1)!\}^{-1/2}(a_1^*)^{m-1}\Omega,\\
s_m(a_{n_1}^*)^{k_1}\cdots(a_{n_p}^*)^{k_p}\Omega =& \ \ \{(m-1)!\}^{-1/2}\,(a_1^*)^{m-1}(a_{n_1+1}^*)^{k_1}\cdots(a_{n_p+1}^*)^{k_p}\Omega,\\
s_m^*\Omega =& \ \ \delta_{m,1}\Omega,\\
s_m^*(a_{n_1}^*)^{k_1}\cdots(a_{n_p}^*)^{k_p}\Omega =& \ \ \begin{cases} \delta_{m,1}(a_{n_1-1}^*)^{k_1}\cdots(a_{n_p-1}^*)^{k_p}\Omega & (n_1\geq 2),\\ \delta_{m,k_1+1}\sqrt{k_1!}\,(a_{n_2-1}^*)^{k_2}\cdots(a_{n_p-1}^*)^{k_p}\Omega & (n_1=1)\end{cases}
\end{aligned}$$

for $1\leq n_1<\cdots<n_p$ and $k_1,\ldots,k_p\in\mathbf{N}$.

Example 3.1 Define the representation $(l_2(\mathbf{N}),\pi)$ of $\mathcal{O}_\infty$ by

$$\pi(s_n)e_m\equiv e_{2^{n-1}(2m-1)}\quad (n,m\in\mathbf{N}). \tag{3.2}$$

Then this is $P_\infty(1)$ with the GP vector e_1. For the representation in (3.2), the vacuum is e_1 and the subspace H_1 of one-particle states is given by

$$H_1\equiv\overline{\mathrm{Lin}\langle\{e_{2^{n-1}+1}:n\geq 1\}\rangle}.$$

We show that the above correspondence holds for $\mathcal{O}_N$ for any $2\leq N<\infty$.

Proposition 3.2 *If we identify $\mathcal{O}_\infty$ with a C^*-subalgebra of $\mathcal{O}_N$ by (2.1), then $P_N(1)|_{\mathcal{B}}=Fock$.*

Proof. Because $P_N(1)|_{\mathcal{O}_\infty}=P_\infty(1)$, $P_N(1)|_{\mathcal{B}}=P_\infty(1)|_{\mathcal{B}}=Fock$. ∎

Let $(\mathcal{H},\pi)$ be $P_N(1)$ of $\mathcal{O}_N$ with the GP vector Ω. From (2.1), the following holds for $1\leq n_1<n_2<\cdots<n_m$ and $k_1,\ldots,k_m\in\mathbf{N}$:

$$(a_{n_1}^*)^{k_1}\cdots(a_{n_m}^*)^{k_m}\Omega=\prod_{i=1}^m\sqrt{k_i!}\;t_1^{n_1-1}t_N^{c_1-1}t_{b_1}T_2\cdots T_m\Omega \tag{3.3}$$

where $T_i\equiv t_1^{n_i-n_{i-1}-1}t_N^{c_i-1}t_{b_i}$ for $i=2,\ldots,m$ and we define $c_i\in\mathbf{N}$ and $b_i\in\{1,\ldots,N-1\}$ by the equation $k_i=(N-1)(c_i-1)+b_i-1$.

Example 3.3 *(Fock representation by $\mathcal{O}_2$ and $\mathcal{O}_3$)* From (3.3), the following holds: When $N = 2$,

$$(a^*_{n_1})^{k_1} \cdots (a^*_{n_m})^{k_m}\Omega = \prod_{i=1}^{m} \sqrt{k_i!}\; t_1^{n_1-1} t_2^{k_1} t_1^{n_2-n_1} t_2^{k_2} \cdots t_1^{n_m-n_{m-1}} t_2^{k_m}\Omega.$$

When $N = 3$,

$$(a^*_{n_1})^{k_1} \cdots (a^*_{n_m})^{k_m}\Omega = \prod_{i=1}^{m} \sqrt{k_i!}\; t_1^{n_1-1} t_3^{c_1-1} t_{b_1} T_2 \cdots T_m\Omega$$

where $T_i \equiv t_1^{n_i-n_{i-1}-1} t_3^{c_i-1} t_{b_i}$ for $i = 2, \ldots, m$ and we define $c_i \in \mathbf{N}$ and $b_i \in \{1, 2\}$ by $k_i = 2(c_i - 1) + b_i - 1$.

3.2 Interpretation of representations by infinite product

In this subsection, we consider representations F_j ($j \geq 2$), F_{12} and F_{21} of bosons in Definition 2.3 from a viewpoint of Fock representation. Formal infinite products of operators are introduced for this purpose.

3.2.1 F_j

For the cyclic vector Ω of F_j in Definition 2.3 with $j \geq 2$, it seems that the formal vector

$$\Omega^{'} \equiv \left(\prod_{n=1}^{\infty} a_n^{j-1} \right) \Omega \qquad (3.4)$$

is a new vacuum of F_j up to normalization constant. The cyclic subspace by $\Omega^{'}$ is equivalent to the Fock representation because $a_n\Omega^{'} = 0$ for each n by formal computation. However such vector can not be defined in the representation space of F_j. Furthermore F_j is not equivalent to the Fock representation F_1 when $j \neq 1$ by Lemma 2.4 (ii). However, the formal notation (3.4) often appears in theoretical physics and it excites curiosity. If we regard that (3.4) is justified by F_j, then (3.4) obtains a meaning of the operation in the representation theory.

3.2.2 F_{12} and F_{21}

According to the case F_j, we write the Fock vacuum by the cyclic vector Ω of F_{12}. Then we obtain the formal vector $\Omega^{'}$ as follows:

$$\Omega^{'} \equiv \left(\prod_{n=1}^{\infty} a_{2n} \right) \Omega. \qquad (3.5)$$

Of course, $\Omega^{'}$ never be defined in the representation space F_{12}.

In the same way, we write the Fock vacuum by the cyclic vector Ω of F_{21}. Then we obtain the formal vector $\Omega^{'}$ as follows:

$$\Omega^{'} \equiv \left(\prod_{n=1}^{\infty} a_{2n-1}\right) \Omega. \tag{3.6}$$

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Appendix

A Proof of Lemma 2.2

(i) Fix $j \geq 1$. We introduce an orthonormal basis of a given representation belonging to $P_\infty(j)$. Let $(\mathcal{H}, \pi)$ be $P_\infty(j)$ with the GP vector Ω. We simply denote $\pi(s_n)$ by s_n for each n. Define the subset Λ_j of $\mathbf{N}^* \equiv \coprod_{l\geq 1} \mathbf{N}^l$ by

$$\Lambda_j \equiv \{(m), J \cup (n), \, n, m \geq 1, \, n \neq j, \, J \in \mathbf{N}^*\}$$

and $v_J \equiv s_J\Omega$ for $J \in \mathbf{N}^*$. Because $s_j\Omega = \Omega$, we see that $\{s_J s_K^*\Omega : J, K \in \mathbf{N}^*\} = \{s_J\Omega : J \in \Lambda_j\}$. Hence $\mathrm{Lin}\langle\{v_J : J \in \Lambda_j\}\rangle$ is dense in $\mathcal{H}$. Furthermore $\langle v_J|\Omega\rangle = 0$ when $J \neq (j)$. This implies that $\langle v_J|v_K\rangle = \delta_{J,K}$ for $J, K \in \Lambda_j$. In consequence $\{v_J : J \in \Lambda_j\}$ is a complete orthonormal basis of $\mathcal{H}$. The construction of $\{v_J : J \in \Lambda_j\}$ is independent of the choice of $\mathcal{H}$ except the existence of GP vector Ω. Hence $P_\infty(j)$ is uniquely up to unitary equivalence.

Assume that $(\mathcal{H}, \pi)$ is $P_\infty(12)$ with the GP vector of Ω. We identify $\pi(s_n)$ with s_n for each n. By definition, we see that $\{s_J\Omega : J \in \mathbf{N}^*\}$ spans a dense subspace of $\mathcal{H}$. Define the sequence $\{T_n \in \mathbf{N}^* : n \in \mathbf{N}\}$ of multiindices by $T_{2k} \equiv (12)^k$ and $T_{2k-1} = (12)^{k-1} \cup (1)$ for each $k \geq 1$. If $J \in \mathbf{N}^n$, then

$$\langle v_J|\Omega\rangle = \delta_{J,T_n}.$$

From this, the orthonormal basis $\{v_J : J \in \Lambda_{12}\}$ of $\mathcal{H}$ is given by

$$v_J \equiv s_J\Omega \quad (J \in \Lambda_{12})$$

where $\Lambda_{12} \equiv \{(n2), (m), J \cup (k), J \cup (l2) : n, m, k, l \in \mathbf{N}, \, k \neq 2, \, l \neq 1, J \in \mathbf{N}^*\}$. Hence the orthonormal basis of $\mathcal{H}$ is determined only by the assumptions of Ω. Hence $P_\infty(12)$ is unique up to unitary equivalence.

(ii) Assume that $P_\infty(i) \sim P_\infty(j)$. Then there exists a representation of $\mathcal{O}_\infty$ with two cyclic vectors Ω and $\Omega^{'}$ satisfying $s_i\Omega = \Omega$ and $s_j\Omega^{'} = \Omega^{'}$. Because $i \neq j$, $\langle \Omega|\Omega^{'}\rangle = 0$. Furthermore we can verify that $\langle v_J|\Omega^{'}\rangle = \delta_{J,(j)^n}\langle \Omega|\Omega^{'}\rangle = 0$ for any $J \in \Lambda_i \cap \mathbf{N}^n$ with respect to the notation in the proof of (i) for i. Hence $\langle v_J|\Omega^{'}\rangle = 0$ for any $J \in \Lambda_i$. This implies that $\Omega^{'} = 0$. This contradicts with the choice of $\Omega^{'}$. Therefore $P_\infty(i) \not\sim P_\infty(j)$.

Fix $i \geq 1$. Assume that $P_\infty(12) \sim P_\infty(i)$. Then there exists a representation of $\mathcal{O}_\infty$ with two cyclic vectors Ω and $\Omega^{'}$ satisfying $s_{12}\Omega = \Omega$ and $s_i\Omega^{'} = \Omega^{'}$. Then $\langle \Omega|\Omega^{'}\rangle = \langle s_{12}\Omega|s_i^2\Omega^{'}\rangle = 0$. For any $J \in \mathbf{N}^n$,

$$\langle v_J|\Omega^{'}\rangle = \delta_{J,(i)^n}\langle \Omega|\Omega^{'}\rangle = 0.$$

This implies $\Omega^{'} = 0$. This contradicts with the choice of $\Omega^{'}$. Hence there exist no such cyclic vector. Therefore the statement holds.
(iii) We use the notation in the proof of (i). Assume that $B \in \mathcal{B}(\mathcal{H})$ satisfies $[B, x] = 0$ for any $x \in \mathcal{O}_\infty$. Then we can verify that $\langle Bv_J|v_K\rangle = \delta_{J,K}\langle B\Omega|\Omega\rangle$ for each $J, K \in \Lambda_j$. This implies that $B = \langle \Omega|B\Omega\rangle \cdot I \in \mathbf{C}I$. Hence the statement holds.

Assume that $\mathcal{O}_\infty$ acts on $\mathcal{H}$ and $\Omega \in \mathcal{H}$ is a cyclic vector such that $s_{12}\Omega = \Omega$. Assume that $B \in \mathcal{B}(\mathcal{H})$ satisfies $[B, x] = 0$ for any $x \in \mathcal{O}_\infty$. Then we can verify that

$$\langle Bv_J|v_K\Omega\rangle = \delta_{JK}\langle B\Omega|\Omega\rangle \quad (J, K \in \Lambda_{12}).$$

From this, $B = \langle \Omega|B\Omega\rangle I \in \mathbf{C}I$. Hence the statement holds. ∎

B Proof of Lemma 2.4

(i) Fix $j \geq 1$. By definition, the following is derived:

$$a_n^j\Omega = 0, \quad a_n^k(a_n^*)^k\Omega = (j + k - 1)\cdots j\Omega \quad (n, k \in \mathbf{N}).$$

In addition, if $j \geq 2$, then the following holds for $1 \leq l \leq j - 1$:

$$(a_n^*)^l a_n^l\Omega = (j - 1)\cdots(j - l)\Omega.$$

If $k \geq j$, then $\langle \Omega|(a_n^*)^k\Omega\rangle = \langle a_n^k\Omega|\Omega\rangle = 0$. If $1 \leq k \leq j - 1$, then

$$\langle \Omega|(a_n^*)^k\Omega\rangle = C\langle \Omega|(a_n^*)^k(a_n^*)^{j-k}a_n^{j-k}\Omega\rangle = C\langle a_n^j\Omega|a_n^{j-k}\Omega\rangle = 0$$

where $C \equiv \{(j-1)\cdots k\}^{-1/2}$. This implies $\langle\Omega|a_n^k\Omega\rangle = 0$ when $1 \leq k \leq j-1$. In consequence,

$$\langle\Omega|a_n^k\Omega\rangle = \langle\Omega|(a_n^*)^k\Omega\rangle = 0 \quad (k, n \geq 1). \tag{B.1}$$

From these, the family of the following vectors is an orthonormal basis of the vector space $\mathcal{B}\Omega$:

$$v = C \cdot (a_{n_1}^*)^{k_1} \cdots (a_{n_p}^*)^{k_p} a_{m_1}^{l_1} \cdots a_{m_q}^{l_q}\Omega \tag{B.2}$$

for $1 \leq n_1 < \cdots < n_p$ and $k_1, \ldots, k_p \in \mathbf{N}$, $1 \leq m_1 < \cdots < m_q$, $l_1, \ldots, l_q \in \{1, \ldots, j-1\}$, $\{n_1, \ldots, n_p\} \cap \{m_1, \ldots, m_q\} = \emptyset$ and $p, q \geq 0$ where we use notations $a_{n_0}^* = a_{m_0} = I$ and

$$C \equiv \left[\prod_{t=1}^{p}\{(j+k_t-1)\cdots j\} \cdot \prod_{r=1}^{q}\{(j-1)\cdots(j-l_r)\}\right]^{-1/2}.$$

In particular, when $j = 1$, we always assume $q = 0$. The existence of the canonical basis consisting of v's in (B.2) implies the uniqueness of the representation. Therefore the statement holds for F_j.

For the cyclic vector Ω of F_{12}, we see that

$$a_{2m-1}^l(a_{2m-1}^*)^l\Omega = l!\,\Omega, \quad a_{2m}^l(a_{2m}^*)^l\Omega = (l+1)!\,\Omega, \quad a_{2m}^2\Omega = 0$$

for $l, m \geq 1$. Define

$$v = C \cdot (a_{2n_1-1}^*)^{k_1} \cdots (a_{2n_p-1}^*)^{k_p}(a_{2m_1}^*)^{l_1} \cdots (a_{2m_q}^*)^{l_q} a_{2t_1} \cdots a_{2t_r}\Omega \tag{B.3}$$

for $1 \leq n_1 < \cdots < n_p$, $1 \leq m_1 < \cdots < m_q$, $1 \leq t_1 < \cdots < t_r$, $\{m_1, \ldots, m_q\} \cap \{t_1, \ldots, t_r\} = \emptyset$ and $k_1, \ldots, k_p, l_1, \ldots, l_q \in \mathbf{N}$ where

$$C = \{k_1! \cdots k_p!\,(l_1+1)! \cdots (l_q+1)!\}^{-1/2}.$$

Then the set of all such v's in (B.3) is an orthonormal basis of $\mathcal{B}\Omega$. Hence the uniqueness of F_{12} holds.

For F_{21}, we can construct an orthonormal basis by replacing the suffixes $2n$ and $2n-1$ in the proof of F_{12}. Hence the uniqueness of F_{21} holds.

(ii) Assume that $i \neq j$ and $F_i \sim F_j$. Then there exists a representation of $\mathcal{B}$ with two cyclic vectors Ω and Ω' satisfying

$$a_n a_n^*\Omega = i\Omega, \quad a_n a_n^*\Omega' = j\Omega' \quad (n \geq 1). \tag{B.4}$$

From this, $\langle\Omega|\Omega^{'}\rangle = 0$. Let

$$x = (a_{n_1}^*)^{k_1}\cdots(a_{n_p}^*)^{k_p}a_{m_1}^{l_1}\cdots a_{m_q}^{l_q} \tag{B.5}$$

for $1 \leq n_1 < \cdots < n_p$, $1 \leq m_1 < \cdots < m_q$, $k_1,\ldots,k_p \in \mathbf{N}$, $l_1,\ldots,l_q \in \{1,\ldots,j-1\}$ and $\{n_1,\ldots,n_p\}\cap\{m_1,\ldots,m_q\} = \emptyset$. Define $M \equiv n_p+m_q+1$. Because $a_M a_M^*$ commutes x, and $i \neq j$, $\langle x\Omega|\Omega^{'}\rangle = 0$ from (B.4). From this and (B.2), $\Omega^{'} = 0$. This contradicts with the choice of $\Omega^{'}$. Hence $F_i \not\sim F_j$ when $i \neq j$.

Assume that $F_j \sim F_{12}$ for some $j \geq 1$. Then there exists a representation of $\mathcal{B}$ with two cyclic vectors Ω and $\Omega^{'}$ satisfying

$$a_n a_n^*\Omega = j\Omega, \quad a_{2n-1}\Omega^{'} = 0, \quad a_{2n}^* a_{2n}\Omega^{'} = \Omega^{'} \quad (n \geq 1). \tag{B.6}$$

From this,

$$a_{2n}a_{2n}^*\Omega^{'} = 2\Omega^{'}, \quad a_{2n-1}a_{2n-1}^*\Omega^{'} = \Omega^{'} \quad (n \geq 1). \tag{B.7}$$

Hence $\langle\Omega|\Omega^{'}\rangle = 0$. Let x be as in (B.5) and $M \equiv n_p+m_q+1$. Because both $a_{2M-1}a_{2M-1}^*$ and $a_{2M}a_{2M}^*$ commute x, $\langle x\Omega|\Omega^{'}\rangle = 0$ from (B.6) and (B.7). This implies $\Omega^{'} = 0$. This contradicts with the choice of $\Omega^{'}$. Therefore $F_j \not\sim F_{12}$ for any $j \geq 1$. In the same way, we see that $F_j \not\sim F_{21}$ for any $j \geq 1$.

Assume $F_{12} \sim F_{21}$. Then there exists a representation of $\mathcal{B}$ with two cyclic vectors Ω and $\Omega^{'}$ satisfying $a_{2n}^*a_{2n}\Omega = \Omega$ and $a_{2n-1}\Omega = 0$ and $a_{2n-1}^*a_{2n-1}\Omega^{'} = \Omega^{'}$ and $a_{2n}\Omega^{'} = 0$ for each $n \geq 1$. Then $\langle\Omega|\Omega^{'}\rangle = \langle a_{2n}^*a_{2n}\Omega|\Omega^{'}\rangle = \langle a_{2n}^*\Omega|a_{2n}\Omega^{'}\rangle = 0$. Let

$$x = (a_{2n_1-1}^*)^{k_1}\cdots(a_{2n_p-1}^*)^{k_p}(a_{2m_1}^*)^{l_1}\cdots(a_{2m_q}^*)^{l_q}a_{2t_1}\cdots a_{2t_r} \tag{B.8}$$

and assume the assumption in (B.3) and $p+q+r \geq 1$. Let $L \equiv 2n_p - 1 + 2m_q + 2t_r + 1$. Then

$$\langle x\Omega|\Omega^{'}\rangle = \langle x a_{2L}^* a_{2L}\Omega|\Omega^{'}\rangle = \langle a_{2L}^* x a_{2L}\Omega|\Omega^{'}\rangle = \langle x a_{2L}\Omega|a_{2L}\Omega^{'}\rangle = 0.$$

This holds for any such x. Hence $\Omega^{'} = 0$. This contradicts with the choice of $\Omega^{'}$. Therefore Assume $F_{12} \not\sim F_{21}$.

(iii) Fix $j \geq 1$. Let Ω be the cyclic vector F_j such that $a_n a_n^*\Omega = j\Omega$ for each $n \in \mathbf{N}$. Assume that $B \in \mathcal{B}(\mathcal{H})$ satisfies $[a_n, B] = [a_n^*, B] = 0$ for each n. Let x be as in (B.5) and assume $p+q \geq 1$. Because B commutes x, $\langle B\Omega|x\Omega\rangle = 0$ from (B.1). This implies that $\langle Bw|z\rangle = 0$ for $w, z \in \mathcal{E}$, $w \neq z$ where $\mathcal{E}$ is the family of v's in (B.2). Therefore the off-diagonal part of B with respect to

vectors in $\mathcal{E}$ is zero. Furthermore, we obtain that $\langle Bv|v\rangle = \langle B\Omega|\Omega\rangle$ for any $v \in \mathcal{E}$. This implies that $B = \langle\Omega|B\Omega\rangle I \in \mathbf{C}I$. Hence F_j is irreducible.

Let $\mathcal{S}$ be the set of all vectors v in (B.3). We see that $\langle\Omega|Bv\rangle = 0$ for $v \in \mathcal{S} \setminus \{\Omega\}$. From this, $\langle v|Bw\rangle = 0$ for $v, w \in \mathcal{S}$, $v \neq w$. Furthermore, from the form of $v \in \mathcal{S}$, we obtain that $\langle v|Bv\rangle = \langle\Omega|B\Omega\rangle$ for any $v \in \mathcal{S}$. Therefore $B = \langle\Omega|B\Omega\rangle I \in \mathbf{C}I$. Hence F_{12} is irreducible. We can prove the irreducibility of F_{21} by replacing the suffixes $2n$ and $2n-1$ in the proof of F_{12}. Hence F_{21} is also irreducible. ∎

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